General Mechanics of Continua

→ An Introduction

refs: - Goldstein, chapter posted

- Whitham } see ref. list
- Lighthill }

- Stone article, posted

- F & W.

f.) More Oscillations : Mechanics of Fields

→ recall the [redacted] string : (i.e. continuum limit)

$$\mathcal{L} = \mathcal{L}(y, y_t, y_x) \quad \rightarrow \text{ Lagrangian density}$$

$$\mathcal{L} = \tfrac{1}{2}\mu\, y_t^2 - \underbrace{T\left[(1+y_x^2)^{1/2} - 1\right]}_{\text{potential energy in string}} \qquad \underline{\underline{(1D!)}}$$

where $S = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L}$ $\qquad$ t, x both parameters

Then, for EOM: $\delta S = 0$ (as usual)

$$\delta S = 0 = \int_{t_1}^{t_2} dt \int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial y_t}\delta y_t + \frac{\partial \mathcal{L}}{\partial y_x}\delta y_x \right)$$

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial y_t}\frac{d}{dt}\delta y + \frac{\partial \mathcal{L}}{\partial y_x}\frac{d}{dx}\delta y \right)$$

$$= \cancel{\int_0^L dx \frac{\partial \mathcal{L}}{\partial y_t}\delta y \Big|_{t_1}^{t_2}} \rightarrow 0 \;+ \int_{t_1}^{t_2} dt \frac{\partial \mathcal{L}}{\partial y_x}\delta y \Big|_0^L$$

$$+ \int_{t_1}^{t_2}\int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right)\delta y$$

fixed end pts in time!
↓
no config change.

thus, have Lagrange EOM :

$$\boxed{\begin{aligned} &\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \\ &\text{with B.C. :} \qquad \frac{\partial \mathcal{L}}{\partial y_x}\,\delta y\Big|_0^L = 0 \end{aligned}}$$

(clear for fixed, free ends)

n.b : → have

- spatial ibp endpt. $\displaystyle \int_{t_1}^{t_2} dt\, \frac{\partial \mathcal{L}}{\partial y_x}\,\delta y \Big|_0^L$

- $\delta y(t, x) = 0$, all x, only at t_2, t_1.

→ in 3D, have:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx_i}\left(\frac{\partial \mathcal{L}}{\partial x_i}\right)$$

→ for 1D string :

$$\frac{d}{dt}\left(\mu\, y_t\right) = \frac{d}{dx}\left(\frac{T\, y_x}{(1+y_x^2)^{1/2}}\right)$$

small oscillations : $\mathcal{L} = \frac{1}{2}\mu y_t^2 - \frac{T}{2}(y_x)^2$

$\therefore$

$\mu y_{t,t} = T y_{xx}$ $\rightarrow$ $\begin{cases}\text{garden variety}\\ \text{wave eqn.}\end{cases}$

$\rightarrow$ Ex. $\quad U(\phi) = \alpha\frac{\phi^2}{2} + \beta\phi^4$

$$\mathcal{L} = \frac{\phi_t^2}{2} - \frac{(\nabla\phi)^2}{2} - U(\phi)$$

Derive

$\Rightarrow$ EOM ? $\Rightarrow$ K-G Eqn.

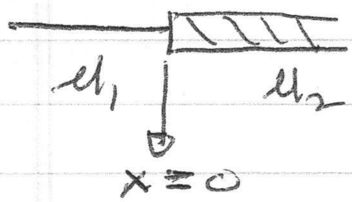
$$\phi_{tt} - \phi_{xx} + \alpha\phi + \beta\phi^3 = 0$$

Aside: Standard Problems

Now, Lagrangian formulation allows unambiguous formulation of basic equations for matching;

$\Rightarrow$ consider $\underline{\underline{2}}$ prototypical examples

How to handle matching conditions?

c.) 

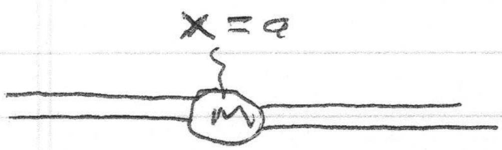
Junction $\Rightarrow$ un-equal mass.

matching $\Rightarrow$ $y_-(0) = y_+(0)$

$$\int_{0_-}^{0_+} \left\{ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \frac{\partial \mathcal{L}}{\partial y} + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right\} = 0$$

i.e. integrate EOM

$$\left.\frac{\partial \mathcal{L}}{\partial y_x}\right|_{0_+} = \left.\frac{\partial \mathcal{L}}{\partial y_x}\right|_{0_-} \Rightarrow \text{slope match}$$

d.) 

(continuity understood)

$\mu \to \mu + M\delta(x-a)$

$$\mathcal{L} = \frac{1}{2}\left(\mu + M\delta(x-a)\right) y_t^2 - \frac{T}{2} y_x^2$$

$$\left(\mu + M\delta(x-a)\right) y_{tt} = T y_{xx}$$

$$y = \tilde{y}(x)\, e^{-i\omega t}$$

4a. ~~[illegible]~~

$$T\hat{y}_{xx} = -\omega^2(\mu + M\delta(x-a))\hat{y}$$

$$\int_{a_-}^{a_+}\left[T\hat{y}_{xx} + \omega^2(\mu + M\delta(x-a))\hat{y}\right] = 0$$

$$T\hat{y}_x \Big|_{a_-}^{a_+} = -\omega^2 M y(a)$$ ⇒ jump condition

⇒ Mass induces jump

N.B.: Use of Lagrangian ab-initio renders all questions re: order of derivatives moot.

→ Hamiltonian Formulation

Define canonical momentum:

$$\pi = \partial \mathcal{L} / \partial \dot{y} \rightarrow \mu \dot{y}$$ → momentum (in y) of string element

$$= \mu \dot{y}$$

Can define Hamiltonian density:

$$\mathcal{H} = \pi \dot{y} - \mathcal{L}$$

$$H = \int_0^L dx\, \mathcal{H}$$

for $\frac{\partial \mathcal{L}}{\partial t} = 0$, $\mathcal{H} = \mathcal{E}$

↳ Hamiltonian.

For string:

$$\mathcal{H} = \frac{\pi^2}{\mu} - \mathcal{L} = \underbrace{\frac{\pi^2}{2\mu}}_{\text{Kin E.}} + \underbrace{\frac{T y_x^2}{2}}_{\text{Pot E}}$$

So, as before, Hamilton's Eqns for continua follow from Principle of Least Action:

$$S = \int_{t_1}^{t_2} dt \int_0^L dx \left(\pi \dot{y} - \mathcal{H}\right)$$

$\mathcal{L} = \mathcal{L}(\dot{y}, y_x, y)$

$\mathcal{H} = \mathcal{H}(\pi, y_x, y)$

$\mathcal{L} = \pi \dot{y} - \mathcal{H}$

So

$$\delta S = \int_{t_1}^{t_2} dt \int_0^L dx \left(\pi \, \delta\dot{y} + \dot{y} \, \delta\pi - \left(\frac{\partial \mathcal{H}}{\partial \pi}\delta\pi + \frac{\partial \mathcal{H}}{\partial y_x}\delta y_x + \frac{\partial \mathcal{H}}{\partial y}\delta y\right)\right)$$

ignoring surface terms:

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left\{ \dot{y} \, \delta\pi - \dot{\pi} \, \delta y - \frac{\partial \mathcal{H}}{\partial y} - \frac{\partial \mathcal{H}}{\partial \pi}\delta\pi - \frac{\partial \mathcal{H}}{\partial y_x}\delta y_x \right\}$$

regrouping

$$\delta S = \int_{t_1}^{t_2} dt \int_0^L dx \left\{ \delta y \left(\frac{d\pi}{dt} + \frac{\partial \mathcal{H}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{H}}{\partial y_x}\right) \right) + \delta\pi \left(\dot{y} - \frac{\partial \mathcal{H}}{\partial \pi} \right) \right\}$$

so $\delta S = 0 \Rightarrow$

$$\begin{cases} \dot{y} = \dfrac{\partial \mathcal{H}}{\partial \pi} \\ \dot{\pi} = -\dfrac{\partial \mathcal{H}}{\partial y} + \dfrac{d}{dx}\left(\dfrac{\partial \mathcal{H}}{\partial y_x}\right) \end{cases}$$

Hamilton's Eqns $\rightarrow$ motion of elements parametrized by x, t.

Now, $\partial \mathcal{L}/\partial t = \partial \mathcal{H}/\partial t = 0$ here

$$\mathcal{H} = \pi \dot{y} - \mathcal{L}$$

$$\frac{d\mathcal{H}}{dt} = \pi \ddot{y} + \dot{y}\dot{\pi} - \frac{d\mathcal{L}}{dt}$$

$$= \pi \ddot{y} + \dot{y}\dot{\pi} - \left(\cancel{\frac{\partial \mathcal{L}}{\partial t}} + \frac{\partial \mathcal{L}}{\partial y}\dot{y} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\ddot{y} + (\partial \mathcal{L}/\partial y_x)\dot{y}_x \right)$$

but, $\pi = \partial \mathcal{L} / \partial \dot{y}$

so $\pi \ddot{y}$ cancels $- \partial \mathcal{L} / \partial \dot{y}) \ddot{y}$

$\Rightarrow$

$$\frac{dH}{dt} = \dot{\pi} \dot{y} - \left(\frac{\partial \mathcal{L}}{\partial y} \dot{y} + \frac{\partial \mathcal{L}}{\partial y_x} \dot{y}_x \right)$$

and from LEOM:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) = \frac{\partial \mathcal{L}}{\partial y}$$

so

$\partial \mathcal{L} / \partial \dot{y} = \pi$

$$\frac{dH}{dt} = \cancel{\dot{\pi} \dot{y}} - \dot{y} \left(\cancel{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)} + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right) - \frac{\partial \mathcal{L}}{\partial y_x} \dot{y}_x$$

$$= - \dot{y} \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) - \frac{\partial \mathcal{L}}{\partial y_x} \frac{d}{dx} \dot{y}$$

$\Rightarrow$ so finally regrouping:

$$\boxed{\frac{d\mathcal{H}}{dt} + \frac{d}{dx}\left(\dot{y}\frac{\partial\mathcal{L}}{\partial y_x}\right) = 0}$$

What does it mean?

$\Rightarrow$ Here $\mathcal{H} = \mathcal{E}$

so above $\Leftrightarrow$

$$\frac{d}{dt}\mathcal{E} + \frac{d}{dx}S_x = 0$$

$\mathcal{E} \rightarrow$ excitation energy density

$S \rightarrow$ excitation energy density flux!

check: $S_x = \dot{y}\frac{\partial\mathcal{L}}{\partial y_x}$

$= -\dot{y}\, y_x T$ $\qquad \omega^2 = c^2k^2$, $c^2 = T/\mu$

$y = A\sin(kx-\omega t)$

$= +A^2 k\omega T\cos^2(kx-\omega t)$

$= k\omega T A^2 \cos^2(kx-\omega t)$

$\omega = ck$

$$\bar{S}_x = \frac{\omega^2}{c} \frac{T A^2}{2} = \frac{\omega^2}{c^2} \frac{T c A^2}{2}$$

phase, grp velocity (dispersionless)

$= c\,\mathcal{E}$ → energy density.

⇒ wave energy density flux

so

$$\left.\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x} S_x = 0\right]$$

$S = c\,\mathcal{E} \;\rightarrow\; v_{gr}\,\mathcal{E}$

Result is a "Poynting Thm." for string

In higher dims:

$$\partial_t \mathcal{E} + \underline{\nabla} \cdot \underline{S} = 0.$$

Note:

→ Poynting thm. relates (local) wave energy density with wave energy density flux, i.e.

$$\frac{d\mathcal{H}}{dt} + \partial_x S_x = 0$$

→ Poynting thm. relates rate of energy change to wave energy density flux thru interval

i.e.

$$\frac{d}{dt} E = \frac{d}{dt}\int_{x_1}^{x_2} \mathcal{H}\, dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} S_x$$

$$= -S_x \Big|_{x_1}^{x_2}$$

→ Poynting thm. formed by expressing $\frac{d}{dt}\mathcal{E}$ as $\underline{\nabla}\cdot\underline{S}$, etc.

recall in E and M:

$$\underline{\nabla}\times\underline{B} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t}$$

$$\underline{\nabla}\times\underline{E} = -\frac{1}{c}\,\partial\underline{B}/\partial t$$

but $\mathcal{E} = E^2/8\pi + B^2/8\pi$

then $\left(\frac{\partial \underline{E}}{\partial t} = c \underline{\nabla} \times \underline{B} - 4\pi \underline{J} \right) * \cdot \underline{E}/4\pi$

$\left(\frac{\partial \underline{B}}{\partial t} = - c \underline{\nabla} \times \underline{E} \right) \cdot (\underline{B}/4\pi)$

→ local power dissipated.
Analogue for string I

$\Rightarrow$

$$\frac{\partial}{\partial t} \left(\frac{E^2 + B^2}{8\pi} \right) = - \underline{E} \cdot \underline{J} - \underline{\nabla} \cdot \underbrace{\left(\frac{c}{4\pi} \underline{E} \times \underline{B} \right)}_{\underline{S}}$$

i.e. form Poynting thm. by considering time rate of change of energy density.

→ Important to distinguish:

(particle)

$\underline{\Pi} = \mu \dot{y} \hat{y} \equiv$ canonical momentum

→ momentum of <u>string element</u> $\mu \dot{y}(x,t)$, in $\hat{y}$ direction

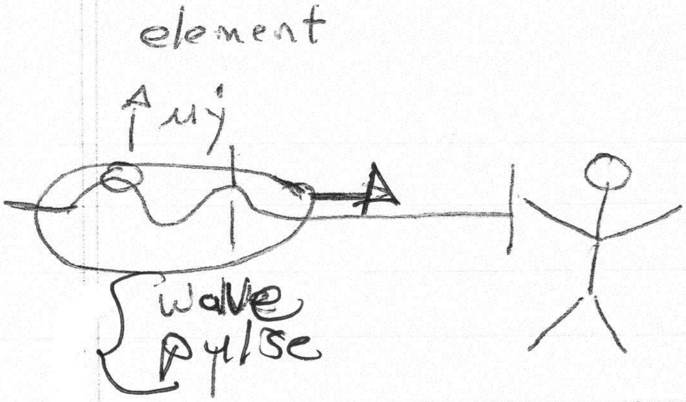
(quasi-particle)

$$\underline{S} = -T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \hat{x} = \frac{\partial \mathcal{L}}{\partial y_x} \frac{\partial y}{\partial t} \hat{x}$$

$\equiv$ wave energy density flux
→ momentum of <u>wave</u>/fluctuation, in $\hat{x}$ direction

13.

i.e.



→ "feels" wave momentum (kick in $\vec{x}$)

flux thru (x) at time t $= \underline{S_x(x,t)}$

calculating for wave on string :

if $y = A\cos(k(x - v_{ph}t))$ $\qquad v_{ph} = (T/\mu)^{1/2}$

$$\frac{\partial y}{\partial t} = +Akv_{ph}\sin(k(x - v_{ph}t))$$

$$\frac{\partial y}{\partial x} = -Ak\sin(k(x - v_{ph}t))$$

$$S_x = +TA^2k^2v_{ph}\sin^2(k(x - v_{ph}t))$$

$$\therefore \quad \overline{S}_x = \frac{Tk^2}{2}v_{ph}A^2$$

but : $\omega^2 = v_{ph}^2k^2$

$$\boxed{\overline{S}_x = \frac{\mu\omega^2}{2}v_{ph}A^2}$$

$\overline{S}_x = v_{gr}\mathcal{E}$

as $v_{gr} = v_{ph}$

$\mathcal{E} = 2 * \overline{KE}$

$= 2 * \frac{1}{4}\mu\omega^2A^2$

→ Wave Momentum Density

- have developed notions of wave energy and Poynting Theorem, ∴

- natural to investigate wave momentum density

Now, recall in E+M,

$$\underline{P}_{EM} = \frac{1}{c^2}\underline{S} = \frac{1}{4\pi c}\underline{E}\times\underline{B} = \frac{1}{c^2}\left(\begin{array}{c}\text{Wave Energy}\\ \text{Density Flux}\end{array}\right)$$

$\underline{P}_{EM}$ ↳ momentum of electromagnetic wave

$\underline{S}$ ↳ Poynting vector

Thus, natural motivation to investigate relation for string, i.e.

$$P = \dot{y}\,\frac{\partial \mathcal{L}}{\partial y_x}$$

so

$$\dot{P} = \ddot{y}\,\frac{\partial \mathcal{L}}{\partial y_x} + \dot{y}\,\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right)$$

for string;

$$\ddot{y} = \frac{T}{\mu}\,y_{xx} = V_{ph}^2\,y_{xx} \quad ; \quad \frac{\partial \mathcal{L}}{\partial y_x} = -T\,y_x$$

$$\Rightarrow \dot{S}_x = \left\{ -\frac{T}{\mu} y_{xx} T \cdot y_x - \mu \dot{y} \frac{T}{\mu} \dot{y}_x \right\}$$

$$= -\frac{T}{\mu} \frac{\partial}{\partial x} \left\{ \frac{T y_x^2}{2} + \frac{\mu \dot{y}^2}{2} \right\}$$

$$= -c^2 \frac{\partial}{\partial x} \mathcal{E}$$

then, have:

$$\frac{\partial}{\partial t} S_x + c^2 \frac{\partial}{\partial x} \mathcal{E} = 0$$

so, if old E+M:

$$c^2 = T/\mu$$

$$P_w = S/c^2$$

wave momentum density

$$\Rightarrow \boxed{\frac{\partial}{\partial t} P_w + \frac{\partial}{\partial x} \mathcal{E} = 0}$$

in 1D

$$\boxed{\frac{\partial}{\partial t} \rho_W + \nabla_x \mathcal{H} = 0}$$

where $\nabla_x \mathcal{H} = \nabla_x \mathcal{E}$ is force density

pushes in direction of propagation

$$\mathcal{P}_W = \int_{x_1}^{x_2} dx \, \rho_W$$

momentum in wave packet in pulse of string $[x_1, x_2]$

so:

$$\frac{\partial}{\partial t} \mathcal{P}_W = -\mathcal{H} \Big|_{x_1}^{x_2}$$

difference / jump in energy density across the chunk of string

$\Rightarrow$ net change in WMD

Note:

a.) semi-classical analogy

$$\mathcal{E} = \omega\, \mathcal{E}/\omega = N\omega$$

↳ wave action density (see next lecture)

$$P_W = \frac{S}{c^2} = \frac{k}{\omega}\frac{\cancel{c}\,\mathcal{E}}{\cancel{c}}$$

$$= \frac{k}{\cancel{\omega}} N \cancel{\omega} = kN$$

Wave energy density → $N\omega$

Wave momentum density → Nk

N → # waves / wave population density.

ii) $\mu \dot{y} = \pi$ → canonical momentum $\hat{y}$ direction

Now symmetry connection:

(a) if string + disturbance translated in $\hat{x}$ and result invariant

$\Rightarrow \exists$ conserved <u>momentum</u>

but

(b) if disturbance/pulse translated on x with string fixed, and result invariant

$\Rightarrow \exists$ conserved Pseudomomentum.

Evidently Pseudomomentum $\Leftrightarrow$

Wave Momentum Density!

(iii) Note can write:

$$\frac{\partial}{\partial t}\mathcal{H} + \frac{\partial}{\partial x} S = 0$$

$$\frac{\partial}{\partial t} P_w + \frac{\partial}{\partial x}\mathcal{H} = 0$$

$$\Rightarrow \left(\frac{1}{v_{ph}}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)\begin{bmatrix} \mathcal{H}, & S/v_{ph} \\ S/v_{ph} & \mathcal{E} \end{bmatrix} = 0$$

i.e. can think as:

$$\partial_\mu T^{\mu\nu} = 0$$

$T^{\mu\nu}$ = energy-momentum tensor of string

$$T^{\mu\nu} = \begin{bmatrix} \mathcal{H}, & S/v_{ph} \\ S/v_{ph}, & \mathcal{H} \end{bmatrix}$$

$$\partial_\mu = \left(1/v_{ph}\, \partial_t,\ \partial_x \right)$$

For EM:

$$T^{ik} = \begin{pmatrix} \underset{\uparrow}{\overset{(E^2+H^2)/8\pi}{W}}, & S_x/c, & S_y/c, & S_z/c \\ S_x/c & & & \\ S_y/c & & \underline{\underline{\sigma}} & \\ S_z/c & & & \end{pmatrix}$$

$$\sigma_{\alpha,\beta} = 1/4\pi \left\{ -E_\alpha E_\beta - H_\alpha H_\beta + \frac{\delta_{\alpha,\beta}}{2} \left(E^2 + H^2 \right) \right\}$$

$\Downarrow$

Maxwell stress tensor.

→ Application : Sound

$$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial}{\partial t} \underline{v} + \underline{v} \cdot \underline{\nabla} \underline{v} = -\frac{\nabla p}{\rho}$$

linearizing ⇒

$$\frac{\partial \tilde{v}}{\partial t} = -\frac{c_s^2}{\rho_0} \underline{\nabla} \tilde{\rho}$$

$$p = p(\rho)$$
$$dp/d\rho = c_s^2$$

$$\frac{\partial}{\partial t} \tilde{\rho} = -\rho_0 \underline{\nabla} \cdot \underline{v}$$

then:

$$\frac{\partial^2}{\partial t^2} \tilde{\rho} = \rho_0 \underline{\nabla} \cdot \left\{ \frac{c_s^2}{\rho_0} \underline{\nabla} \tilde{\rho} \right\} = c_s^2 \nabla^2 \tilde{\rho}$$

$$\partial_t^2 \tilde{\rho} = c_s^2 \nabla^2 \tilde{\rho}$$

⇒ wave eqn.

then: $\frac{\partial^2 \hat{\rho}}{\partial t^2} = c_s^2 \nabla^2 \hat{\rho} = \rho_0 \underline{\nabla} \cdot \left\{ \frac{c_s^2}{\rho_0} \underline{\nabla} \rho \right\}$

For energy-momentum relations:

$(1) \cdot \hat{V} \rho_0 + (2) \cdot \frac{\hat{\rho} c_s^2}{\rho_0} \Rightarrow$

$$\frac{\partial}{\partial t}\left(\frac{\rho_0 \hat{V}^2}{2}\right) + c_s^2 \, \hat{\underline{V}} \cdot \nabla \hat{\rho} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\hat{\rho}^2 c_s^2}{2\rho_0}\right) + c_s^2 \hat{\rho} \, \underline{\nabla} \cdot \hat{\underline{V}} = 0$$

$$\therefore \quad \frac{\partial}{\partial t}\left(\frac{\rho_0 \hat{V}^2}{2} + \frac{\hat{\rho}^2 c_s^2}{2\rho_0}\right) + \underline{\nabla} \cdot \left[c_s^2 \rho \underline{V} \right] = 0$$

$$H = \mathcal{E} = \underbrace{\frac{\rho_0 \hat{V}^2}{2}}_{T} + \underbrace{\frac{\hat{\rho}^2 c_s^2}{2\rho_0}}_{V} \qquad \underline{S} \rightarrow \text{acoustic wave energy density flux}$$

$T \rightarrow$ fluid motion

V (compression)

Similarly,

$$\underline{P}_w = \frac{1}{c_s^2} \underline{S}$$

$$\frac{\partial P_w}{\partial t} = \frac{\partial}{\partial t}(\rho \underline{V}) = \frac{\partial \hat{\rho}}{\partial t} \hat{\underline{V}} + \hat{\rho} \frac{\partial \underline{V}}{\partial t}$$

$$\Rightarrow \quad \frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \underline{\nabla} \cdot \underline{v} \qquad (1)$$

$$\frac{\partial \tilde{v}}{\partial t} = -\frac{c_s^2}{\rho_0} \underline{\nabla} \tilde{\rho} \qquad (2)$$

$$\underline{v}\ (1) + \tilde{\rho}\ (2) \Rightarrow$$

$$\frac{\partial}{\partial t}(\tilde{\rho} \underline{\tilde{v}}) = -\rho_0 \underline{v}(\underline{\nabla} \cdot \underline{v}) - \frac{c_s^2}{2\rho_0} \nabla(\tilde{\rho}^2)$$

$$= -\underline{\nabla}\left(\frac{\rho \tilde{v}^2}{2} + \frac{c_s^2}{\rho_0}\frac{\tilde{\rho}^2}{2}\right)$$

[Momentum density relation for longitudinal linear waves.

$$\frac{\partial}{\partial t}\frac{\underline{S}}{c_s^2} = -\underline{\nabla}\ \underline{\Sigma}$$

$$= \frac{\partial}{\partial t} \underline{P}_w$$

{ ignores all but linear wave energy